Math1105 Midterm Test - Sample

S. Rodney, Cape Breton University Mathematics

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Solutions.

This is a closed book test. There are **5** problems worth **34** points in total. The value of each problem is given in brackets before each problem statement. You may not use notes of any kind nor communicate with any persons with the exceptions of Dr. Rodney and his assistant(s). Note that the CBU policy concerning academic dishonesty will be enforced. Communication devices are prohibited as are graphing calculators. You are permitted to use a standard scientific calculator. If you have a question, raise your hand and someone will be with you shortly.

Good Luck!

1. **True/False Problems** Circle **T** for true and **F** for false. You do not need to show your work. Each item is worth 1 mark.

(a)	If $\lim_{x\to 3} f(x) = 5$ then it must be that $f(3) = 5$.	**T**	**F**
(b)	If $f(x)$ is differentiable at x, then f may not be continuous at x.	**T**	**F**
(c)	The function $f(x) = \sin\left(\frac{\pi}{x}\right)$ has no limit at $x_0 = 0$ because it has a vertical asymptote at $x_0 = 0$.	**T**	**F**
(d)	$f(x) = \dfrac{3x^3+1}{12x^3-1}$ has the horizontal asymptote $y = 1/4$.	**T**	**F**

2. **Some differentiation**. Find the derivative of each of the following. Show all of your work.

   (a) (3) $f(x) = \dfrac{x-1}{x+1}$.

   $$f'(x) = \frac{(x+1)D(x-1) - (x-1)D(x+1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

   (b) (3) $g(t) = 2t^{\frac{3}{2}} - 2t^{-\frac{3}{2}}$.

   $$g'(t) = 3t^{1/2} + 3t^{-5/2}$$

3. **Fun with the Derivative.**

   (a) (4) Use the **definition of the derivative** to find $g'(2)$ if $g(x) = 2\sqrt{x}$.

   (b) (2) Using your answer above, find the equation of the tangent line to $g(x)$ at $x = 2$.

4. **Some special methods.**

   (a) (3) Show that the function $f(x) = x^3 + 6x^2 + 81x - 7$ has a root and then find an interval of length 1 that contains the root.

   (b) (3) Let $f(x) = 4x^3 - 20x$. Find all values of x where the tangent line to $f(x)$ has slope 4.

5. **Some Limits!** Evaluate each of the following limits or indicate if the do not exist. If the limit does not exist, explain why. Show all of your work.

   (a) (3) $\displaystyle\lim_{x\to 2} \frac{\sqrt{2x^2-4}-2}{x-2}$

   (b) (3) $\displaystyle\lim_{z\to 3} \frac{|z-3|}{z-3}$.

   (c) (3) $\displaystyle\lim_{x\to\infty} \frac{\sin^2(x)}{x}$.

6. **Fun with Graphs!**

   (a) (2) Draw the graph of a function $f(x)$ that is continuous on $(-\infty, \infty)$ that is not differentiable.

   (b) (3) Consider the function defined on $(-\infty, \infty)$:

   $$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

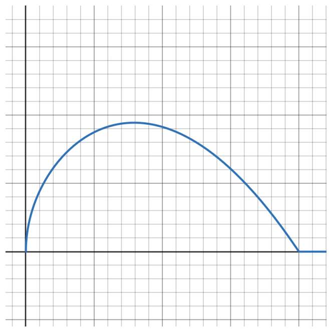
      i. (2) Sketch the graph on f as best you can.

      ii. (1) Is $f(x)$ continuous at $x = 0$? Justify your answer.

7. **An Application:** (3) Suppose a projectile is fired at a battlefield with height given by $p(t) = t^{\frac{1}{2}} - \frac{1}{8}t^2$ Km where t is in seconds and $t = 0$ corresponds to when the cannon fires (see graph below). At what time does the

cannonball reach its maximum height? (Hint: When the projectile is at its maximum height, $p(t)$ has a horizontal tangent!)

8. (BONUS +1) When does the projectile hit the battlefield?



#3. (a) $\lim_{h\to 0} \frac{g(2+h)-g(2)}{h}$

$= \lim_{h\to 0} \frac{2\sqrt{2+h}-2\sqrt{2}}{h}$

$= \lim_{h\to 0} \frac{2(2+h-2)}{h(\sqrt{2+h}+\sqrt{2})}$

$= \lim_{h\to 0} \frac{2}{\sqrt{2+h}+\sqrt{2}}$ So, $g'(2) = \frac{1}{\sqrt{2}}$

$= \frac{2}{2\sqrt{2}}$

$= 1/\sqrt{2}$

(b) The slope of our tangent is $g'(2) = \frac{1}{\sqrt{2}}$ and so it's equation has the form

$$y = \frac{1}{\sqrt{2}}x + b.$$

Since $(2, g(2)) = (2, 2\sqrt{2})$ is on our line,

$2\sqrt{2} = \frac{2}{\sqrt{2}} + b = \sqrt{2} + b$ as $\frac{2}{\sqrt{2}} = \sqrt{2}$.

giving $b = \sqrt{2}$.

Our tangent is

$$y = \frac{1}{\sqrt{2}}x + \sqrt{2}.$$

#4. (a) $p(x) = x^3 + 6x^2 + 81x - 7$. Notice that $p(0) = -7 < 0$ and $p(1) = 81 > 0$. Since p is a poly$^{\underline{n}}$ it's continuous on $\mathbb{R} = (-\infty, \infty)$ and so, also on $[0,1]$. So, the IVT gives a value $c \in (0,1)$ s.t.

$p(c) = 0$. * $(0,1)$ has length 1 since $1-0 = 1$.

b) $f(x) = 4x^3 - 20x$. The slope of the tangent to f at x is $f'(x) = 12x^2 - 20$. Since we are looking for tangents with slope 4, we solve

$$f'(x) = 12x^2 - 20 = 4$$

$$\text{i.e. } 12x^2 = 24$$

$$\text{i.e. } x^2 = 2 \longrightarrow x = \pm\sqrt{2}$$

So, f has tangents of slope 4 at $x = \pm\sqrt{2}$

#5. (a) $\displaystyle\lim_{x\to 2} \frac{\sqrt{2x^2-4} - 2}{x-2}$

if $x \neq 2$, $\displaystyle\frac{\sqrt{2x^2-4}-2}{x-2} = \frac{2x^2-4-4}{(x-2)(\sqrt{2x^2-4}+2)}$

$$= \frac{2(x-2)(x+2)}{(x-2)(\sqrt{2x^2-4}+2)}$$

So, our limit is $\displaystyle\frac{2(4)}{\sqrt{4}+2} = \frac{8}{4} = 2.$

(b) $\displaystyle\lim_{z\to 3} \frac{|z-3|}{z-3}$. This limit does not exist.

to see why, notice that

$$\lim_{z\to 3^+} \frac{|z-3|}{z-3} = \lim_{z\to 3^+} \frac{z-3}{z-3} = 1,$$

and $\displaystyle\lim_{z\to 3^-} \frac{|z-3|}{z-3} = \lim_{z\to 3} \frac{3-z}{z-3} = -1$ with $1 \neq -1$.

(c) $\displaystyle\lim_{x\to\infty} \frac{\sin^2(x)}{x}$.

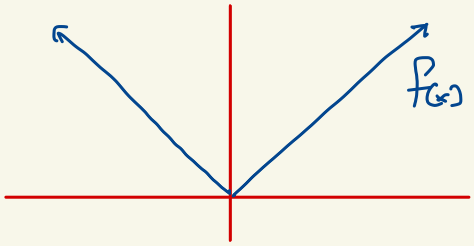
Recall that $-1 \leq \sin(x) \leq 1$ for every $x \in \mathbb{R}$
and so $0 \leq \sin^2(x) \leq 1$ for every $x \in \mathbb{R}$ too.
More, if $x > 0$ we also find

$$0 \leq \frac{\sin^2(x)}{x} \leq \frac{1}{x}.$$

Since $\displaystyle\lim_{x\to\infty} \frac{1}{x} = 0$, the Squeeze thm gives

$$\lim_{x\to\infty} \frac{\sin^2(x)}{x} = 0.$$

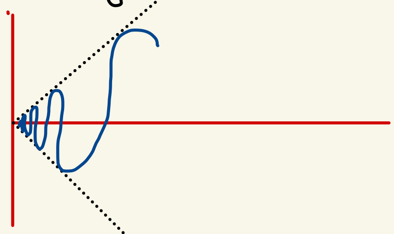
#6(a) $f(x) = |x|$;



$f(x)$ is not diff^{ble} at $x=0$

*there are lots more examples! try to be creative.

(b) $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right); & x>0 \\ 0 & ; x=0 \end{cases}$



- f is continuous at $x=0$ since

$$\lim_{x\to0^+} x\sin\left(\frac{1}{x}\right) = 0 = f(0).$$

* since $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for any $x>0$, we find

$$-x \le x\sin\left(\frac{1}{x}\right) \le x \text{ for } x>0 \text{ too.}$$

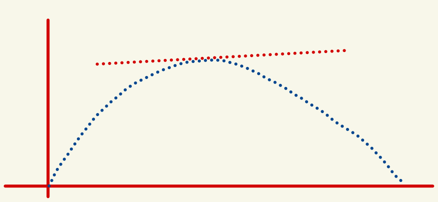
Since $\lim_{x\to0^+}(-x) = 0 = \lim_{x\to0^+}(x)$

the squeeze thm proves our claim.

#7. $p(t) = t^{1/2} - \frac{1}{8}t^2$

i) The projectile reaches its max height when the tangent to $p(t)$ is flat.

$\frac{4}{3} - 3$
$= -\frac{5}{3}$



$2^{1/3} - \frac{1}{8}(2^{4/3})$

$2^{1/3} - 2^{-5/3}$

Now, $p'(t) = \frac{1}{2}t^{-1/2} - \frac{1}{4}t$. So, $p(t) = 0$

if $\frac{1}{2}t^{-1/2} = \frac{1}{4}t$ i.e. $t^{3/2} = 2$ giving $t = 2^{2/3}$.

So, the max height is reached at $t = \sqrt[3]{4}$.

ii) The projectile hits the ground when $p(t) = 0$

i.e. $t^{1/2} = \frac{1}{8}t^2$

$\rightarrow t^{3/2} = 8 \rightarrow t = 8^{2/3} = 4$ seconds.