Solutions to Math 1105 Sample Midterm

1. **True / False**

   (a) **F**. Example: $f(x) = x^2$ on $[0, 1]$ has $f'(0) = 0$ but $f(0) \neq f(1)$.

   (b) **F**. Derivatives of $\cos x$ repeat every 4: $\cos x, -\sin x, -\cos x, \sin x$. Since $713 \equiv 1 \pmod 4$, $f^{(713)}(x) = -\sin x$.

   (c) **F**. $\arcsin(x)$ is one-to-one on $[-1, 1]$.

   (d) **T**. $\dfrac{dy}{dx} = Ky$ has general solution $y = Ce^{Kx}$ for any C.

2. **Differentiation**

   (a) $f(x) = \sec x \, e^{2x}$:
   $$f'(x) = e^{2x} \sec x (2 + \tan x).$$

   (b) $g(t) = \sqrt{-\ln(\cos t)}$:
   $$g'(t) = \frac{\tan t}{2\sqrt{-\ln(\cos t)}}.$$

   (c) $h(x) = \dfrac{\arctan x}{x + 1}$:
   $$h'(x) = \frac{(x+1)/(1+x^2) - \arctan x}{(x+1)^2}.$$

3. **Mean Value Theorem**

   Let $f(x) = x^2 \sin(\pi x) + (1 - x)$.

   (a) f is continuous and differentiable everywhere, so the MVT applies on any interval $[a, b]$.

   (b) $f'(x) = 2x \sin(\pi x) + \pi x^2 \cos(\pi x) - 1$. Since $f(1) = f(b) = 0$ for some $b > 1$, by Rolle's Theorem there is $c \in (1, b)$ with $f'(c) = 0$.

   (c) Let $g(x) = |x|$ on $[-1, 1]$. Then $g(-1) = g(1) = 1$, but $g'(x) = \pm 1$ for $x \neq 0$ and undefined at 0. The MVT fails because g is not differentiable on the open interval.

4. **Implicit Differentiation**

   (i) For $x^2 + 4x + 4y^2 = 12$:
   $$2x + 4 + 8yy' = 0 \quad \Rightarrow \quad y' = -\frac{x+2}{4y}.$$

   Horizontal tangents $\Rightarrow y' = 0 \Rightarrow x = -2$. Substitute: $(-2)^2 + 4(-2) + 4y^2 = 12 \Rightarrow y = \pm 2$.
   $$\boxed{(-2, 2), (-2, -2)}.$$

(ii) Let $y = \sin^{-1}(t)$ so $\sin y = t$. Then $\cos y \dfrac{dy}{dt} = 1$, giving

$$\frac{dy}{dt} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-t^2}}.$$

5. **Higher Derivatives**

$$s(t) = bt^3 + ct^2 + dt + e, \quad v(t) = 3bt^2 + 2ct + d, \quad a(t) = 6bt + 2c, \quad j(t) = 6b.$$

Given:

$$s(1) = 1, \quad v(1) = 3, \quad a(1) = -6, \quad j(1) = 12.$$

$$b = 2, \quad c = -9, \quad d = 15, \quad e = -7.$$

$$s(t) = 2t^3 - 9t^2 + 15t - 7.$$

Acceleration $a(t) = 12t - 18$:

$$a(t) > 0 \text{ for } t > \tfrac{3}{2} \text{ (velocity increasing)}, \quad a(t) < 0 \text{ for } t < \tfrac{3}{2} \text{ (velocity decreasing)}.$$

6. **Logarithmic Differentiation**

(a) $y = (x+1)(x^2+1)^2(x^3+1)^3(x^4+1)^4(x^5+1)^5$

$$\ln y = \ln(x+1) + 2\ln(x^2+1) + 3\ln(x^3+1) + 4\ln(x^4+1) + 5\ln(x^5+1).$$

$$\frac{y'}{y} = \frac{1}{x+1} + \frac{4x}{x^2+1} + \frac{9x^2}{x^3+1} + \frac{16x^3}{x^4+1} + \frac{25x^4}{x^5+1}.$$

$$\boxed{y' = y\left[\frac{1}{x+1} + \frac{4x}{x^2+1} + \frac{9x^2}{x^3+1} + \frac{16x^3}{x^4+1} + \frac{25x^4}{x^5+1}\right]}.$$

(b) $f(x) = (x+1)^{x+1}$:

$$\ln y = (x+1)\ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + 1 \Rightarrow f'(x) = (x+1)^{x+1}(\ln(x+1)+1).$$

The usual power and exponential rules fail because both base and exponent depend on x.

7. **Exponential Decay**

$$C(t) = C_0 e^{kt}, \quad A_0 = 25, \quad A(5) = 19.61.$$

$$e^{5k} = \frac{19.61}{25} = 0.7844 \Rightarrow k = \frac{\ln(0.7844)}{5} \approx -0.0486.$$

Half-life:

$$T_{1/2} = \frac{\ln(1/2)}{k} \approx 14.27 \text{ yr.}$$

So the element is **Plutonium-241**.

To reach 1 kg:

$$1 = 25\left(\frac{1}{2}\right)^{t/14.29} \Rightarrow \frac{t}{14.29} = \frac{\ln(1/25)}{\ln(1/2)} = \frac{\ln 25}{\ln 2} \approx 4.64.$$

$$\boxed{t \approx 66.4 \text{ years.}}$$

8. **Related Rates**

Let horizontal distance $x = 30$ cm. Snails at heights $y_1 = 25t$, $y_2 = 15t$. Distance between them:

$$D^2 = 30^2 + (y_1 - y_2)^2.$$

Differentiate:

$$2D\frac{dD}{dt} = 2(y_1 - y_2)(y_1' - y_2').$$

$$\frac{dD}{dt} = \frac{(y_1 - y_2)(y_1' - y_2')}{D}.$$

At $y_1 = 100$, $t = 4$, so $y_2 = 60$, $\Delta y = 40$, $D = \sqrt{30^2 + 40^2} = 50$, and $y_1' - y_2' = 10$:

$$\boxed{\frac{dD}{dt} = \frac{40}{50} \times 10 = 8 \text{ cm/min.}}$$