Homework #5 Solutions

#1. (a) By Geometric Series, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ whenever $|x|<1$. Differentiating term by term,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1) x^{n-2}$$

$$\vdots$$

$$\frac{24}{(1-x)^5} = \sum_{n=4}^{\infty} n(n-1)(n-2)(n-3) x^{n-4}$$

$$= \sum_{n=0}^{\infty} (n+4)(n+3)(n+2)(n+1) x^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+4)!}{n!} x^n$$

and we find $\frac{1}{(1-x)^5} = \sum_{n=0}^{\infty} \frac{(n+4)!}{24 n!} x^n$

From discussion in class the IoC does not change from our original geo. series $I = (-1,1)$.

$\underline{\underline{\text{OR}}}$ $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+5)!}{24(n+1)!} \Big/ \frac{(n+4)!}{24n!}$

$$= \lim_{n\to\infty} \frac{(n+5)!}{(n+4)!} \cdot \frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty} \frac{n+5}{n+1}$$

$$= 1$$

So, IoC is $I = (-1, 1)$.

(B) $f(x) = (1-x)\ln|1-x| + 2x$.

Geo-Series, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

Integrating,

$$-\ln|1-x| + C = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}x^{n+1}\right)$$

Since our series converges at $x=0$,
we see $-\ln(1) + C = 0$

$$\to C = 0$$

and so

$$\ln|1-x| = -\left(\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}\right)$$

So, $(1-x)\ln|1-x| + 2x$

$$= -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+2} + 2x$$

$$= -x - \sum_{n=1}^{\infty} \frac{1}{n+1} x^{n+1} + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+2} + 2x$$

$$= -\sum_{n=0}^{\infty} \frac{1}{n+2} x^{n+2} + \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+2} + x$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) x^{n+2} + x.$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} x^{n+2} + x.$$

$$= x^2 \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} x^{n} + x.$$

again, the IOC is $(-1,1)$ since this was true for the geometric series $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

ALTERNATIVE: $f(x) = (1-x)\ln|1-x| + 2x$

$$f'(x) = -\ln|1-x| - (1-x)\cdot\frac{1}{1-x} + 2$$

$$= 1 - \ln|1-x|.$$

$$f''(x) = \frac{1}{1-x}$$

Since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$; $|x|<1$

$$f'(x) = \int f''(x)\,dx = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} + C.$$

Since $f'(0) = 1$, we must have $C = 1$ giving

$$f'(x) = \left(\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}\right) + 1 = \left(\sum_{n=1}^{\infty} \frac{1}{n} x^n\right) + 1$$

finally, $f(x) = \int f'(x)\,dx = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} x^{n+1} + x + d$

Since $f(0) = 0$, we must have $d = 0$ giving

$$f(x) = \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} + x$$

$$= \sum_{n=2}^{\infty} \frac{x^n}{(n-1)\cdot n} + x$$

The interval of convergence is the same as for $\sum_{n=0}^{\infty} x^n$; $(-1,1)$.

OR This expansion converges exactly when

$\sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} x^n$ converges. Now,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)(n+2)}{(n+2)(n+3)} = 1$$

and the Radius of conv is $R=1$

and the IOC is $I=(-1,1)$.

#2. (a) $f(x) = \cos(x)$ $\qquad f(0)=1$

$f'(x) = -\sin(x)$ $\qquad f'(0)=0$

$f''(x) = -\cos(x)$ $\qquad f''(0)=-1$

$f'''(x) = \sin(x)$ $\qquad f'''(0)=0$

$f^{iv}(x) = \cos(x)$ $\qquad f^{iv}(0)=1$

$\vdots$ $\qquad \vdots$

So, $$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

- $$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} x^{2n+2}}{(2n+2)!}}{\frac{(-1)^n x^{2n}}{(2n)!}} \right|$$

$$= |x|^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0,$$

So, the R.o.C is $R = \frac{1}{L} = \infty$ and our interval of convergence is $(-\infty, \infty)$.

(b) $g(x) = \ln(1+x)$ $\qquad g(0) = 0$

$g'(x) = \frac{1}{1+x}$ $\qquad g'(0) = 1$

$g''(x) = \frac{-1}{(1+x)^2}$ $\qquad g''(0) = -1$

$g'''(x) = \frac{2}{(1+x)^3}$ $\qquad g'''(0) = 2!$

$$g^{iv}(x) = \frac{-3!}{(1+x)^4} \qquad g^{iv}(0) = -3!$$

$$g^{v}(x) = \frac{4!}{(1+x)^5} \qquad g^{v}(0) = 4!$$

$$\vdots$$

So, $$\sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!}x^n = g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \frac{g'''(0)}{3!}x^3 + \cdots$$

$$= 0 + x - \frac{x^2}{2!} + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 + \frac{4!}{5!}x^5 \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}$$

Next, $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \right| = 1$

and so RoC is $R = 1/L = 1$ and IoC is

$I = (-1, 1)$.

#3. a) $\ln\left(\frac{3}{2}\right) = \ln\left(1 + \frac{1}{2}\right)$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \frac{1}{2^n}$$

So, $\ln\left(\frac{3}{2}\right) \approx \sum_{n=1}^{k} \frac{(-1)^{n+1}}{n2^n}$.

Setting $k = 6$, $\sum_{n=1}^{6} \frac{(-1)^{n+1}}{n2^n} = \frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64} + \frac{1}{5 \cdot 32} - \frac{1}{6 \cdot 64}$

$$\simeq -0.4046875$$

Calculator: $\ln(1.5) = -0.4054651$

$* \quad |0.4046875 - 0.4054651| < 0.0008 < 10^{-3}$.

(b) $\cos\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{2^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{4^n}$

So, for any k,

$$\cos(1/2) \simeq \sum_{n=0}^{k} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{4^n}$$

Using $k = 5$,

$$\cos(1/2) = \sum_{n=0}^{5} \frac{(-1)^n}{(2n)!} \cdot \frac{1}{4^n}$$

$$= 1 - \frac{1}{8} + \frac{1}{4! \cdot 16} - \frac{1}{6! \cdot 64} + \frac{1}{8! \cdot 4^8} - \frac{1}{10! \cdot 4^{10}}$$

$$\cong 0.87758256$$

Calculator:

$$\cos(0.5) \approx 0.877582561$$

these values are the same and so the error between the calculator and our estimate is $0 < 10^{-3}$.