Math 1206 Class Test #2 - sample

Solutions

Name:

Student Number:

(1) (a) Find the Taylor expansion of $f(x) = \ln(x)$ centred at $x_0 = 1$.

(b) Find the Maclaurin expansion for $g(x) = xe^x$.

(c) Using your result from (a), use the third partial sum of the series you found to estimate $\ln(1/2)$. How accurate is your answer in comparison to the calculator?

(a) $f(x) = \ln(x)$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1 = 0!$

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(1) = -1!$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(1) = 2!$

$f^{iv}(x) = -\frac{3!}{x^4}$ $\qquad f^{iv}(1) = -3!$

$\vdots \qquad \vdots$

Our Taylor Series is

$$\sum_{n=1}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + f'''(1)(x-1)^3 + \cdots$$

$$= (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n \quad //$$

(b) $g(x) = xe^x$; $g(0) = 0$

$g'(x) = (x+1)e^x$; $g'(0) = 1$

$g''(x) = (x+2)e^x$; $g''(0) = 2$

$g'''(x) = (x+3)e^x$; $g'''(0) = 3$

$\vdots$

$g^{(n)}(x) = (x+n)e^x$; $g^{(n)}(0) = n$

Our Maclaurin Series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)x^2}{2} + \frac{f'''(0)x^3}{3!} + \cdots$$

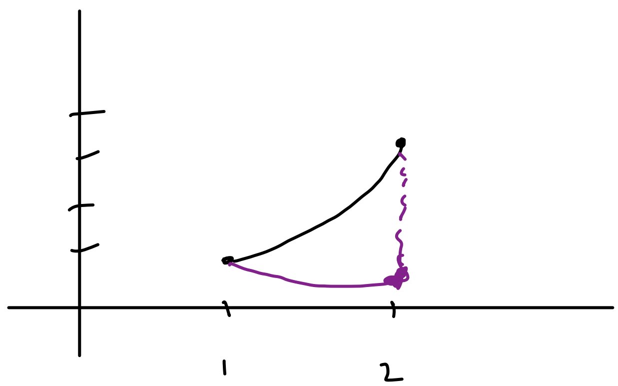
$$= x + \frac{2}{2!}x^2 + \frac{3}{3!}x^3 + \frac{4}{4!}x^4 + \frac{5}{5!}x^5 + \cdots$$

$$= x + \frac{x^2}{1!} + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!}$$

(C) $S_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$ $\quad = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!}$

$S_3(1/2) = -1/2 - \frac{1}{2}(-1/2)^2 + \frac{1}{3}(-1/2)^3 \simeq -0.6667$; Calc: $\ln(1/2) \simeq -0.693$; error is $\simeq 0.026$.

(2) Consider the region lying between and the two curves $f(x) = x^2$ and $g(x) = \frac{1}{x^2}$ over the interval $[1, 2]$. Find the area of this region.



$$A = \int_1^2 |f(x) - g(x)|\,dx = \int_1^2 x^2 - \frac{1}{x^2}\,dx$$

$$= \frac{x^3}{3} + \frac{1}{x} \Big|_1^2$$

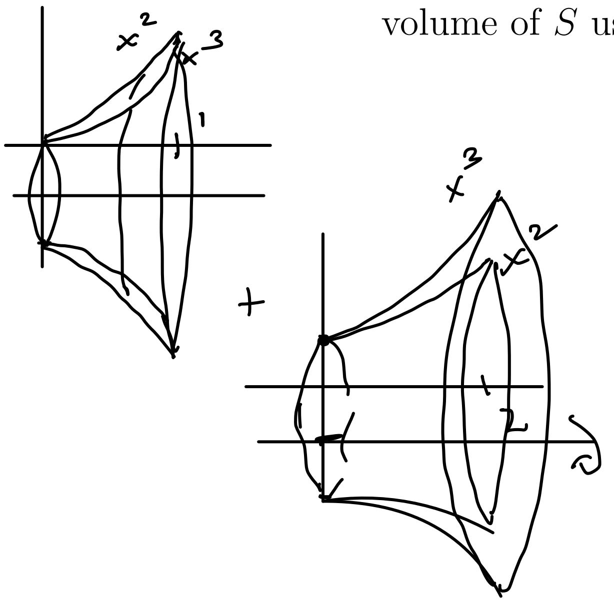
$$= \frac{8}{3} + \frac{1}{2} - \frac{1}{3} - 1$$

$$= \frac{19}{6} - \frac{4}{3}$$

$$= \frac{11}{6} \text{ sq. units.}$$

2 parts.

(3) Consider the solid S obtained by spinning the region between x^2 and x^3 over $[0, 2]$ about the horizontal line $y = -1$. Find the volume of S using the washer method.



$$V = \pi \int_0^2 R^2 - r^2 \, dx$$

$$= \pi \left[\int_0^1 \left((1+x^2)^2 - (1+x^3)^2 \right) dx + \int_1^2 (1+x^3)^2 - (1+x^2)^2 \, dx \right]$$

$$= \pi \left[\int_0^1 (x^4 - 2x^3 + 2x^2 - x^6) \, dx + \int_1^2 (x^6 - x^4 + 2x^3 - 2x^2) \, dx \right]$$

$$= \pi \left[\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{2}{3}x^3 - \frac{1}{7}x^7 \Big|_0^1 + \left(\frac{x^7}{7} - \frac{x^5}{5} + \frac{1}{2}x^4 - \frac{2}{3}x^3 \Big|_1^2 \right) \right]$$

$$= \pi \left[\frac{1}{5} - \frac{1}{2} + \frac{2}{3} - \frac{1}{7} + \frac{2^7}{7} - \frac{2^5}{5} + \frac{1}{2}2^4 - \frac{16}{3} - \frac{1}{7} + \frac{1}{5} - \frac{1}{2} + \frac{2}{3} \right]$$

$$= \pi \left(\frac{47}{210} + \frac{3103}{210} \right) = 15\pi \text{ cubic units.}$$

(4) You have designed a table leg created by spinning the graph of $\frac{1}{9}x^3$ about the x-axis over the interval $[1/2, 3]$ with all measurements in inches. If paint costs .75 per square inch, how much will you need to spend on paint per leg? (You may assume one coat of paint is sufficient)

Surface area is $2\pi \int_{1/2}^{3} \frac{1}{9}x^3 \sqrt{1+\frac{1}{9}x^4}\,dx$

$$= \frac{2\pi}{27} \int_{1/2}^{3} x^3 \sqrt{9+x^4}\,dx$$

$u = 9+x^4$

$\frac{1}{4}du = x^3 dx$

$$= \frac{2\pi}{108} \int_{145/16}^{90} \sqrt{u}\,du$$

$$= \frac{4\pi}{324} u^{3/2} \Big|_{145/16}^{90}$$

$$= \frac{4\pi}{324} \left(90^{3/2} - \left(\frac{145}{16}\right)^{3/2} \right)$$

$\simeq 32.06$ sq. units.

So, our painting cost is

$\simeq 32.06 \cdot 0.75$

$= \$24.03$ per leg.

(5) Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{1}{\sqrt{9x^2-36x+37}}\,dx$ (Hint: Complete the square under the root.)

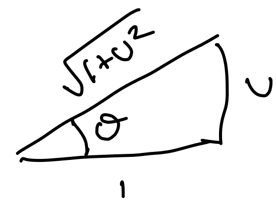
(b) $\displaystyle\int \frac{\sqrt{x^2+16}}{x^4}\,dx$

(a) $9x^2-36x+37 = (3x-6)^2+1$

So, our integral is

$$\int \frac{1}{\sqrt{(3x-6)^2+1}}\,dx \; ; \quad u = 3x-6, \quad \tfrac{1}{3}du = dx$$

$$= \frac{1}{3}\int \frac{1}{\sqrt{u^2+1}}\,du \; ; \quad \text{set } u = \tan(\theta), \quad du = \sec^2(\theta)\,d\theta$$



$$= \frac{1}{3}\int \frac{\sec^2(\theta)}{\sec(\theta)}\,d\theta$$

$$= \frac{1}{3}\int \sec(\theta)\,d\theta = \frac{1}{3}\ln|\sec(\theta)+\tan(\theta)| + C$$

$$= \frac{1}{3}\ln\left|\sqrt{1+u^2}+u\right| + C$$

$$= \frac{1}{3}\ln\left|\sqrt{9x^2-36x+37}+3x+6\right| + C.$$

b) $x = 4\tan(\theta)$

$dx = 4\sec^2(\theta)\,d\theta$

$$\int \frac{\sqrt{16+x^2}}{x^4}\,dx = \int \frac{\sqrt{16+16\tan^2(\theta)}}{\tan^4(\theta)} \cdot 4\sec^2(\theta)\,d\theta$$

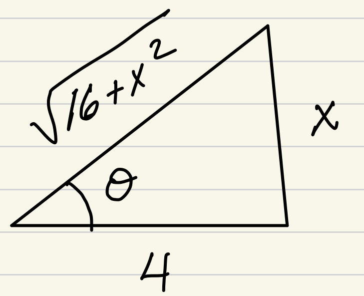
$$= \frac{16}{4^4} \int \frac{\sec(\theta)\sec^2(\theta)}{\tan^4(\theta)} d\theta$$

$$= \frac{16}{4^4} \int \frac{\cos^4(\theta)}{\sin^4(\theta)} \cdot \frac{1}{\cos^3(\theta)} d\theta$$

$$= \frac{16}{4^4} \int \frac{\cos(\theta)}{\sin^4(\theta)} d\theta$$

$u = \sin\theta$
$du = \cos(\theta) d\theta$

$$= \frac{16}{4^4} \int u^{-4} du$$



$$\csc(\theta) = \frac{\sqrt{16+x^2}}{x}$$

$$= -\frac{1}{16 \cdot 3} u^{-3} + C$$

$$= \frac{-1}{16 \cdot 3} \csc^3(\theta) + C$$

$$= \frac{-1}{48} \frac{(16+x^2)^{3/2}}{x^3} + C$$

(6) (a) Find a partial fraction decomposition for $f(x) = \dfrac{3x^2+4x+4}{x(x+1)^2}$.

(b) Using your answer above, find $\int f(x)\,dx$.

(a) $$\frac{3x^2+4x+4}{x(x+1)^2} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)+Bx}{x(x+1)} + \frac{C}{(x+1)^2}$$

$$= \frac{(A(x+1)+Bx)(x+1) + Cx}{x(x+1)^2}$$

So $3x^2+4x+4 = (A(x+1)+Bx)(x+1)+Cx$

- if $x=0$, $4=A$
- if $x=-1$, $-C=3 \to C=-3$
- if $x=1$, $11 = (2A+B)(2)+C$

$$= 16+2B-3$$
$$= 13+2B \quad \text{i.e. } 2B=-2 \to B=-1$$

So, our PFD is

$$\frac{3x^2+4x+4}{x(x+1)^2} = \frac{4}{x} - \frac{1}{(x+1)} - \frac{3}{(x+1)^2}$$

b) $\int \frac{3x^2+4x+4}{x(x+1)^2}\,dx = 4\ln|x| - \ln|x+1| + \frac{3}{x+1} + C.$