Math 1206 Class Test #1 - Practice Test

This is meant for practice and so there are more problems here than on your actual test. Your test will consist of 7 problems only and, each problem will have fewer parts.

(1) Evaluate each of the following integrals and indicate the technique you use if applicable.

(a) $\int \left(x^{-3/2} + 3x^{\pi} - 6\right)\, dx$ $= -2x^{-1/2} + \frac{3}{\pi+1}x^{\pi+1} - 6x + C.$

(b) $\int t^3 \ln(t^4+1)\, dt$

(c) $\int (s+1)e^s\, ds$

(d) $\int \sin^2(s)\cos^3(s)\, ds$

(e) $\int \frac{1}{x\,(\ln(x))^4}\, dx$

(f) $\int \frac{y}{4+3y}\, dy$

(g) $\int \left(1 - \frac{u^2}{1+u^2}\right)\, du$

(b) $u = t^4+1$ $\quad \frac{1}{4}du = t^3 dt \rightarrow \frac{1}{4}\int \ln(u)\, du$ $\quad w = \ln(u)$ $\quad dv = dx$ $\quad dw = \frac{1}{u}dx$ $\quad v = u$

$= \frac{1}{4}\left(u\ln(u) - \int du\right)$

$= \frac{1}{4}\left(u\ln(u) - u\right) + C$

$= \frac{1}{4}(t^4+1)\ln(t^4+1) - \frac{t^4}{4} + C$

(c) Set $u = s+1$ $\quad dv = e^s ds$ $\quad du = ds$ $\quad v = e^s$ $\rightarrow (s+1)e^s - \int e^s ds$

$= (s+1)e^s - e^s + C$

$= se^s + C$

(d) Let $u = \sin(s)$ $\quad du = \cos(s)ds$ $\quad * \cos^2(s) = 1 - \sin^2(s)$

$\int \sin^2(s)\cos^3(s)ds = \int \sin^2(s)(1-\sin^2(x))\cos(x)dx$

$$= \int u^2(1-u^2)\,du$$

$$= \int (u^2 - u^4)\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= \frac{1}{3}\sin^3(s) - \frac{1}{5}\sin^5(x) + C.$$

(e) $\displaystyle\int \frac{1}{x(\ln(x))^4}\,dx$ Let $u = \ln(x)$

$du = \frac{1}{x}\,dx \rightarrow \displaystyle\int u^{-4}\,du$

$$= -\frac{1}{3}u^{-3} + C$$

$$= -\frac{1}{3}(\ln(x))^{-3} + C$$

(f) $\displaystyle\int \frac{y}{4+3y}\,dy$ Let $\omega = 4+3y$

$\frac{1}{3}d\omega = dy$ $\quad * \; y = \frac{1}{3}(\omega - 4)$

$$= \frac{1}{9}\int \frac{\omega - 4}{\omega}\,d\omega = \frac{1}{9}\int\left(1 - \frac{4}{\omega}\right)d\omega$$

$$= \frac{1}{9}\omega - \frac{4}{9}\ln|\omega| + C$$

$$= \frac{1}{3}y - \frac{4}{9}\ln|4+3y| + C.$$

(g) $\displaystyle\int\left(1 - \frac{u^2}{1+u^2}\right)du = \int \frac{1}{1+u^2}\,du$

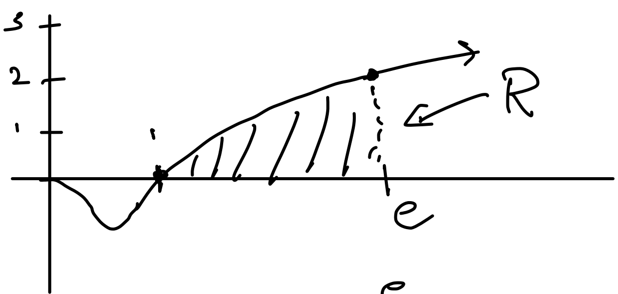
$$= \tan^{-1}(u) + C.$$

(2) Draw a picture of the region R below the function $f(x)$ and above the indicated interval on the x-axis. Following this, calculate the area of R.

(a) $f(x) = x\ln(x)$ over $[1, e]$.

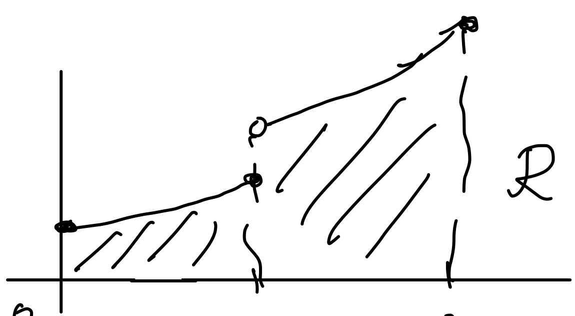
(b) $f(x) = \begin{cases} e^x & \text{if} \quad 0 \le x \le 1 \\ e^x + 1 & \text{if} \quad 1 < x \le 2 \end{cases}$ over $[1, 2]$. $[0,2]$

(a)



$$A(R) = \int_1^e x\ln(x)\,dx \qquad u = \ln(x) \quad dv = x\,dx$$
$$du = \frac{1}{x}dx \quad v = \frac{x^2}{2}$$
$$= \frac{x^2}{2}\ln(x)\Big|_1^e - \frac{1}{4}x^2\Big|_1^e$$
$$= \frac{e^2}{2} - 0 - \frac{1}{4}e^2 + \frac{1}{4}$$
$$= \frac{1}{4}e^2 + \frac{1}{4} = \frac{1}{4}(e^2+1).$$

b)



$$A(R) = \int_0^1 e^x\,dx + \int_1^2 (1+e^x)\,dx$$
$$= e^x\Big|_0^1 + (x+e^x)\Big|_1^2 = e - 1 + 2 + e^2 - 1 - e$$
$$= e^2.$$

(3) Consider $f(x) = \tan(x)\sec^2(x)$ on the interval $[0, \pi/4]$ and define $g(s) = \int_0^s f(x)\,dx$.

(a) Using the substitution $u = \tan(x)$, evaluate $g(s)$.
(b) Using the substitution $u = \sec(x)$, evaluate $g(s)$.
(c) Are your results the same? If not, explain why this is this case.

(a) $\int \tan(s)\sec^2(s)\,ds$ $u = \tan(x)$

$du = \sec^2(x)\,dx$

$= \int u\,du$

$= \frac{u^2}{2} + c = \tan^2(x) + c$

(b) $u = \sec(x)$

$du = \sec(x)\tan(x)\,dx$

$= \int u\,du = \frac{u^2}{2} + c = \sec^2(x) + c$

(c) No, but they differ by only a constant since for any c, x,

$\tan^2(x) + c = \sec^2(x) - 1 + c$ since $\tan^2(x) + 1 = \sec^2(x)$

(4) Evaluate each of the following definite integrals.

(a) $\int_{1}^{e^2} x(\ln(x))^2 \, dx$

(b) $\int_0^{\infty} \frac{e^x}{1+e^{2x}} \, dx$

(c) $\int_0^1 \frac{1}{\sqrt{x}} \, dx$

(a) $u = (\ln(x))^2 \quad dv = x\,dx$

$du = \frac{2\ln(x)}{x} \quad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}(\ln(x))^2 \Big|_1^{e^2} - \int_1^{e^2} x\ln(x)\,dx \qquad w = \ln(x) \quad dv = x\,dx$$

$$dw = \frac{1}{x}dx \quad v = \frac{x^2}{2}$$

$$= \frac{e^4}{2} - \left(\frac{x^2}{2}\ln(x)\Big|_1^{e^2} - \frac{1}{4}x^2\Big|_1^{e^2}\right)$$

$$= \frac{e^4}{2} - \left(e^4 - \frac{1}{4}e^4 + \frac{1}{4}\right) = -\frac{1}{4}e^4 + \frac{1}{4}.$$

(b) $\int_0^{\infty} \frac{e^x}{1+e^{2x}}\,dx \qquad u = e^x$

$du = e^x dx \,; \quad x = 0 \to u = 1$

$x = c \to u = e^c$

$$= \lim_{c\to\infty} \int_0^c \frac{e^x}{1+e^{2x}}\,dx = \lim_{c\to\infty} \int_1^{e^c} \frac{1}{1+u^2}\,du$$

$$= \lim_{c\to\infty} \left(\tan^{-1}(e^c) - \tan^{-1}(1)\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

(c) $\int_0^1 x^{-1/2}\,dx = \lim_{c\to 0^+} \int_c^1 x^{-1/2}\,dx$

$= \lim_{c\to 0^+} 2x^{1/2}\Big|_c^1$

$= \lim_{c\to 0^+} \left(2-2\sqrt{c}\right) = 2.$

(5) Consider the sequence $\left\{\frac{n^2}{\frac{1}{2}+n^4}\right\}_{n=1}^{\infty}$.

(a) Write the first 4 terms of the sequence.
(b) Is the sequence increasing or decreasing? Justify.
(c) Does the sequence converge? If so, to what limit?
(d) Decide if $\sum_{n=1}^{\infty} \frac{n^2}{\frac{1}{2}+n^4}$ converges or diverges. Justify.

(a) $a_1 = \frac{1}{\frac{1}{2}+1} = \frac{2}{3}$, $a_2 = \frac{4}{\frac{1}{2}+16} = \frac{8}{33}$

$a_3 = \frac{9}{\frac{1}{2}+81} = \frac{18}{163}$, $a_4 = \frac{16}{\frac{1}{2}+256} = \frac{32}{513}$.

(b) Set $f(x) = \frac{x^2}{\frac{1}{2}+x^4}$. Then,

$$f'(x) = \frac{2x\left(\frac{1}{2}+x^4\right) - x^2(4x^3)}{\left(\frac{1}{2}+x^4\right)^2} = \frac{x + 2x^5 - 4x^5}{\left(\frac{1}{2}+x^4\right)^2}$$

$$= \frac{x(1-2x^4)}{\left(\frac{1}{2}+x^4\right)^2} < 0 \text{ for } x \geq 1.$$

So, $\left\{\frac{n^2}{\frac{1}{2}+n^4}\right\}_{n=1}^{\infty}$ is decreasing.

(c) $\lim_{n\to\infty} \frac{n^2}{\frac{1}{2}+n^4} = 0$

(d) Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent p-series ($p=2$),

and we see $\frac{n^2}{\frac{1}{2}+n^4} \leq \frac{n^2}{n^4} = \frac{1}{n^2}$ for each n, the comparison test shows $\sum_{n=1}^{\infty} \frac{n^2}{\frac{1}{2}+n^4}$ converges.

(6) Consider the sequence $\left\{\frac{e^n-1}{e^n}\right\}_{n=1}^{\infty}$.

(a) Decide if the sequence is increasing or decreasing.

(b) Is the sequence bounded above or below?

(c) Using a theorem from class, decide if the sequence converges or diverges without evaluating a limit.

(a) $\frac{e^n-1}{e^n} = 1 - \frac{1}{e^n}$. Since e^n is increasing in n, $\frac{1}{e^n}$ is decreasing and so, $1-\frac{1}{e^n}$ is increasing.

(b) $0 < 1 - \frac{1}{e^n} < 1$ for all n since $0 < \frac{1}{e^n} \le \frac{1}{e} < \frac{1}{2}$ and so $0 \le \frac{1}{2} < 1 - \frac{1}{e^n} < 1$. So, the seq. is bounded.

(c) By the Monotone Convergence Theorem, $\left\{\frac{1-e^n}{e^n}\right\}_{n=1}^{\infty}$ converges.

(7) (a) Find the sum of the series $\sum_{n=1}^{\infty}\left[\left(\frac{2}{7}\right)^n - \frac{2}{7}\left(\frac{2}{7}\right)^n\right]$.

(b) Find a formula for the partial sum

$$S_k = \sum_{n=1}^{k}\left[\frac{1}{n} - \frac{1}{n+1}\right].$$

Using the formula you found, find $\sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+1}\right]$.

(a) $\sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n$ converges since its Geometric with $r=\frac{2}{7}\in(-1,1)$. As a consequence, so does $\sum_{n=1}^{\infty}\frac{2}{7}\left(\frac{2}{7}\right)^n$. So, by the Geo series formula:

$$\sum_{n=1}^{\infty}\left(\left(\frac{2}{7}\right)^n - \frac{2}{7}\left(\frac{2}{7}\right)^n\right) = \sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n - \frac{2}{7}\sum_{n=1}^{\infty}\left(\frac{2}{7}\right)^n$$

$$= \frac{2/7}{1-2/7} - \frac{2}{7}\,\frac{2/7}{1-2/7}$$

$$= \frac{2}{5} - \frac{2}{7}\cdot\frac{2}{5}$$

$$= \frac{2}{5} - \frac{4}{35} = \frac{10}{35} = \frac{2}{7}.$$

OR our series is $\sum_{n=1}^{\infty}\left(\left(\frac{2}{7}\right)^n - \left(\frac{2}{7}\right)^{n+1}\right) = \lim_{k\to\infty}\sum_{n=1}^{k}\left(\left(\frac{2}{7}\right)^n - \left(\frac{2}{7}\right)^{n+1}\right)$

$$= \lim_{k\to\infty}\left(\frac{2}{7} - \left(\frac{2}{7}\right)^{k+1}\right)$$

$$= 2/7.$$

(b) $S_k = \sum_{n=1}^{k}\left(\frac{1}{n}-\frac{1}{n+1}\right) = 1-\frac{1}{k+1}$ (telescoping)

So, $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{k\to\infty}\left(1-\frac{1}{k+1}\right) = 1.$

(8) Using convergence tests, decide which of the following series converge or diverge.
**Indicate the test you use.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{4+n^6}$.

(b) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$.

(a) Comparison with $\sum_{n=1}^{\infty}\frac{1}{n^3}$ ← convergent 3-series.

for each n, $\frac{n^3}{4+n^6} \le \frac{n^3}{n^6} = \frac{1}{n^3}$.

(b) $\displaystyle\int_1^{\infty} xe^{-x}\,dx = \lim_{c\to\infty}\int_1^c xe^{-x}\,dx$ $\quad u=x \quad dv=e^{-x}dx$

$\quad du=dx \quad v=-e^{-x}$

$$= \lim_{c\to\infty}\left(-xe^{-x}\Big|_1^c + \int_1^c e^{-x}\,dx\right)$$

$$= \lim_{c\to\infty}\left(e^{-1} - ce^{-c} - e^{-c} + e^{-1}\right)$$

$$= \lim_{c\to\infty}\left(2e^{-1} - (c+1)e^{-c}\right)$$

$$= 2e^{-1} < \infty$$

Since the integral converges, so does $\sum_{n=1}^{\infty} ne^{-n}$ by the integral test!