1206 HW #3 Solution.

#1. (i) $\displaystyle\int_{\pi/2}^{\pi} \frac{14\tan(x)\cos^2(x)}{1+\sin^2(x)}\,dx$

$= 14\displaystyle\int_{\pi/2}^{\pi} \frac{\sin(x)\cos(x)}{1+\sin^2(x)}\,dx$

Let $u = 1+\sin^2(x)$

$du = 2\sin(x)\cos(x)\,dx$

$\longrightarrow \frac{1}{2}du = \sin(x)\cos(x)\,dx$

* $x = \pi/2 \longrightarrow u = 2$

$x = \pi \longrightarrow u = 1$

$= 7\displaystyle\int_{2}^{1} \frac{1}{u}\,du$

$= 7\ln|u|\Big|_2^1$

$= -7\ln 2$

$= \ln\left(2^{-7}\right).$

(ii) $\displaystyle\int_{1}^{e} \frac{\sqrt{\ln(x)}}{x}\,dx$

Let $u = \ln(x)$

$du = \frac{1}{x}\,dx$

* $x = 1 \longrightarrow u = 0$

$x = e \longrightarrow u = 1$

$= \displaystyle\int_{0}^{1} u^{1/2}\,du$

$$= \frac{2}{3} u^{3/2} \Big|_0^1 = 2/3.$$

(iii) $\displaystyle\int_1^2 \frac{1+x^3+x^4+x^5}{x}\,dx$

$$= \int_1^2 \frac{1}{x}\,dx + \int_1^2 x^2\,dx + \int_1^2 x^3\,dx + \int_1^2 x^4\,dx$$

$$= \ln|x| \Big|_1^2 + \frac{1}{3}x^3 \Big|_1^2 + \frac{1}{4}x^4 \Big|_1^2 + \frac{1}{5}x^5 \Big|_1^2$$

$$= \ln(2) + \frac{2^3}{3} - \frac{1}{3} + \frac{2^4}{4} - \frac{1}{4} + \frac{2^5}{5} - \frac{1}{5}.$$

$$= \ln(2) + 2^3\left(\frac{1}{3} + \frac{1}{2} + \frac{4}{5}\right) - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right)$$

$$= \ln(2) + 8 \cdot \frac{49}{30} - \frac{47}{60} = \ln(2) + \frac{737}{60}.$$

#2. (i) $\displaystyle\int x e^{\pi x}\,dx \qquad u = x \qquad dv = e^{\pi x}dx$

$$du = dx \qquad v = \frac{1}{\pi} e^{\pi x}$$

$$= \frac{1}{\pi} x e^{\pi x} - \frac{1}{\pi}\int e^{\pi x}\,dx$$

$$= \frac{1}{\pi} x e^{\pi x} - \frac{1}{\pi^2} e^{\pi x} + C$$

$$= \frac{1}{\pi}\left(x - \frac{1}{\pi}\right)e^{\pi x} + C.$$

(ii) $\int t^{72}\ln(t)\,dt$; $u = \ln(t)$ $dv = t^{72}dt$

$du = \frac{1}{t}dt$ $v = \frac{1}{73}t^{73}$

$$= \frac{1}{73}t^{73}\ln(t) - \frac{1}{73}\int t^{72}dt$$

$$= \frac{1}{73}t^{73}\left(\ln(t) - \frac{1}{73}\right) + C.$$

(iii) $I = \int e^{x}\cos(2x)\,dx$ Let $u = \cos(2x)$ $dv = e^{x}dx$

$du = -2\sin(2x)$ $v = e^{x}$

$$= e^{x}\cos(2x) + 2\int e^{x}\sin(2x)\,dx$$ Let

$w = \sin(2x)$ $dv = e^{x}dx$

$dw = 2\cos(2x)dx$ $v = e^{x}$

$$= e^{x}\cos(2x) + 2\left(e^{x}\sin(2x) - 2I\right)$$

and we find

$$5I = e^{x}\left(\cos(2x) + 2\sin(2x)\right)$$

and so, $I = \frac{1}{5}e^{x}\left(\cos(2x) + 2\sin(2x)\right) + C$

#3. (i) $\int_0^\infty \frac{x}{(x^2+1)^3}\,dx = \lim_{a\to\infty} \int_0^a \frac{x}{(x^2+1)^3}\,dx$ (*)

Let $u = x^2+1$ if $x=0$, $u=1$

$\frac{1}{2}du = x\,dx$ if $x=a$, $u=a^2+1$

So, (*) $= \frac{1}{2}\lim_{a\to\infty} \int_1^{a^2+1} u^{-3}\,du$

$$= -\frac{1}{4}\lim_{a\to\infty} \frac{1}{u^2}\Big|_1^{a^2+1}$$

$$= -\frac{1}{4}\lim_{a\to\infty}\left(\frac{1}{(a^2+1)^2} - 1\right)$$

$= \frac{1}{4}$ since $\lim_{a\to\infty} \frac{1}{(a^2+1)^2} = 0$.

(ii) $\int_1^2 \frac{1}{\sqrt{x-1}}\,dx$

$= \lim_{s\to 1^+} \int_s^2 \frac{1}{\sqrt{x-1}}\,dx$; $u = x-1$ $x=2 \to u=1$

$du = dx$; $x=s$, $u=s-1$

$= \lim_{s\to 1^+} \int_{s-1}^1 u^{-1/2}\,du$

$$= \lim_{s \to 1^+} 2u^{1/2} \Big|_{s-1}^{1}$$

$$= \lim_{s \to 1^+} 2\left(\sqrt{1} - \sqrt{s-1}\right)$$

$$= 2 \text{ since } \lim_{s \to 1^+} \sqrt{s-1} = 0.$$

(iii) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{b \to -\infty} \int_{b}^{0} \frac{1}{1+x^2}\,dx + \lim_{a \to \infty} \int_{0}^{a} \frac{1}{1+x^2}\,dx$$

$$= \lim_{b \to -\infty} \tan^{-1}(x)\Big|_{b}^{0} + \lim_{a \to \infty} \tan^{-1}(x)\Big|_{0}^{a}$$

$$= \lim_{b \to -\infty} \left(\frac{\pi}{4} - \tan^{-1}(b)\right) + \lim_{a \to \infty} \left(\tan^{-1}(a) - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \frac{\pi}{4}$$

$$= 2\left(\frac{\pi}{2}\right)$$

$$= \pi.$$